

On Maximization of  $\prod_{cyc}(a - b)$  on the basis of  
given parameters

Sayan Mukherjee, Kolkata

June 2, 2012

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# 1 Introduction

Most inequalities that we come across have a simple type of equality case, ie  $a = b = c$  or  $a = b, c = 0$ . But when we think of maximizing the product  $(a - b)(b - c)(c - a)$  or its magnitude (both mean the same), the equality case cannot be as simple as  $(1, 1, 1)$  because any two equal values of  $a$  and  $b$  can change the value of this expression directly to zero. Of course, considering the same thing over nonnegative reals and whole of reals is much different. So these types of inequalities usually have different equality cases which are not so obvious to determine. However, we will demonstrate how most of these inequalities can be easily solved with the help of AM-GM Inequality. After that, we will try to determine some general forms of this maximum, for different fixed parameters.

## 1.1 Pre-requisites

Here we assume that the reader has a good idea of the different ways of applying the AM-GM inequality, and the Cauchy-Schwarz inequality. These are the most basic and classical inequalities having monstrous applications. Also some basic knowledge about the  $pqr$  and  $uvw$  methods might come in handy, so that the reader can relate between these stuff easily. Basic knowledge on calculus, derivatives and solving polynomial equations is also required.

## 2 The $|(a - b)(b - c)(c - a)|$ problem

We will discuss this method by considering a well-known problem of Tran Quoc Anh (forum name, Nguoivn) which runs as follows.

- If  $a, b, c \geq 0$ , then we have

$$(a + b + c)^3 \geq 6\sqrt{3}|(a - b)(b - c)(c - a)|.$$

### 2.1 Determining the Equality Case

Firstly, we have to order  $a, b, c$  so as to remove the modulus sign. Also, note that in these inequalities, it's important to figure out an equality case somehow. In most of the cases, the equality occurs when one of the variables is set to zero. This is due to the nature of  $(c - a)(c - b)(b - a)$  (assuming  $c \geq b \geq a$ .) When  $a$  is the minimum of the three numbers, we see that the expression itself is a decreasing function in terms of  $a$ , if we keep  $b, c$  fixed. So this expression will be able to reach its maximum for a given value of  $b, c$  if and only if  $a$  is as small as possible, which is zero itself. Plugging in  $a = 0$  and  $c = bt$  leads us to the cubic equation

$$(t + 1)^3 = 6\sqrt{3}t(t - 1).$$

A factorisation of this cubic expression gives us

$$(t - \sqrt{3} - 2)^2 (t + 7 - 4\sqrt{3}) = 0;$$

Which leads us to the positive **double root**  $t = \sqrt{3} + 2$ .

### 2.2 Significance of the Double Root

The inequality that we get after putting  $a = 0$  under the assumption  $c \geq b \geq a$  is

$$(b + c)^3 \geq 6\sqrt{3}bc(c - b).$$

Note that in this inequality, when the equality occurs for some value of  $\frac{c}{b}$ , the curve  $(t + 1)^3 - 6\sqrt{3}t(t - 1)$  must be just touching the  $t$  axis at that point, so as to satisfy the inequality. Another consequence of this concept is that the other root will be on the negative side of the  $t$  axis. Now, we can differentiate this equation to get the double root easily.

$$\left[ (t + 1)^3 - 6\sqrt{3}t(t - 1) \right]' = 3(t + 1)^2 - 12\sqrt{3}t - 1 = 3(t - 2 - \sqrt{3})(t + 4 - 3\sqrt{3})$$

Now, we can check that which one of these is our double root by just plugging in these values into the previous equation.

### 2.3 The Equality Case

If we see that  $c = (\sqrt{3} + 2)b$  satisfies the conditions of equality, then the actual equality case turns out to be

$$(a, b, c) = (0, 1, (2 + \sqrt{3})) \equiv (0, \sqrt{3} - 1, \sqrt{3} + 1)$$

The writing as conjugate surds is for aesthetic satisfaction.

### 2.4 Determining the Coefficients for AM-GM

Once we have determined the equality case, the rest is somewhat easy. Assuming that  $c \geq b \geq a$  helps us writing

$$|(a - b)(b - c)(c - a)| \leq bc(c - b);$$

And  $(a + b + c) \geq (b + c)$ ; So that it is sufficient to check that

$$(b + c)^3 \geq 6\sqrt{3}bc(c - b).$$

For this, we need to apply AM-GM in such a way that the three terms we choose on the right hand side are balanced. Using the equality case, we see that

$$(\sqrt{3} + 1)b = (\sqrt{3} - 1)c = c - b = 2,$$

And therefore the terms have been determined.

$$2bc(c - b) = [(\sqrt{3} + 1)b] [(\sqrt{3} - 1)c] [(c - b)] \leq \frac{\{\sqrt{3}(b + c)\}^3}{27} = \frac{(b + c)^3}{3\sqrt{3}}.$$

Since we have determined the equality case, therefore we need not worry about the right hand side. This method works mostly without fail, and is easy to implement. So, the method has worked out for  $6\sqrt{3}$ . We will discuss some more applications of this method. However, note that higher the degree of the inequality, the more unusable this method becomes.

### 3 Determination of Best Constants

In cases, these  $|(a-b)(b-c)(c-a)|$  problems come with some determination of the "best constant" phrases. What it actually means is, say, we were talking of the inequality

$$(a+b+c)^3 \geq k|(a-b)(b-c)(c-a)|.$$

Where  $k$  is some positive real number, and  $a, b, c \geq 0$ . In such a problem, they want us to figure out the largest value that  $k$  can attain, because the inequality implies  $k \leq \frac{(a+b+c)^3}{|(a-b)(b-c)(c-a)|}$ . If this is satisfied for all  $a, b, c \geq 0$ , then we might try to minimize the right side first by minimizing  $a$ , ie putting  $a = 0$  and  $c = tb$  where  $t \geq 1$ . Then this rewrites into

$$k \leq \frac{(t+1)^3}{t(t-1)}.$$

This is a single-variable inequality, so we may just want to differentiate this to find the minimum of  $f(t) = \frac{(t+1)^3}{t(t-1)}$ . Simple calculus gives us the value  $k = 6\sqrt{3}$  for  $t = 2 + \sqrt{3}$ . This way can also help us determine the equality case as we saw in the previous section. For some applications, we can get our hands on some problems. Note that this is, in cases, not complete enough to prove the original problem. There may be some other parametrization in the problem, for example, an extra  $(ab+bc+ca)$  term may be there, thus affecting the nature of the  $|(a-b)(b-c)(c-a)|$  term, which was always decreasing in  $\min\{a, b, c\}$ . But, this can always fetch us the best constants, and the equality cases in a packed fashion, so that the application of AM-GM becomes easier.<sup>1</sup> So, let's continue with some fifth degree inequalities involving the  $|(a-b)(b-c)(c-a)|$  term.

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<sup>1</sup>**Caution:** In an exam, we never mention the differentiations etc, just scribble the AM-GM step. All of this should be rough work.

## 4 Fifth Degree Inequalities

### 4.1 Problem 1.

For nonnegative reals  $a, b, c$ , find the best  $k$  satisfying

$$(a + b + c)^5 \geq k(ab + bc + ac)(a - b)(b - c)(c - a)$$

*Tran Quoc Anh, from "Topic of Nguoivn"*

#### 4.1.1 Determination of $k$ and other coefficients

Put  $a = 0$  and  $c = bt$  without any hesitation to get  $(t + 1)^5 \geq kt^2(t - 1)$ ; and quickly consider  $f(t) = \frac{(t + 1)^5}{t^2(t - 1)}$ . We see that

$$f'(t) = \frac{2(t + 1)^4(t^2 - 3t + 1)}{t^3(t - 1)^2};$$

Which becomes zero at  $t = \frac{3 \pm \sqrt{5}}{2}$ . Plugging this in leads to  $k = 25\sqrt{5}$ , and the equality case is  $(a, b, c) = (0, \sqrt{5} - 1, \sqrt{5} + 1)$ .

#### 4.1.2 Main Proof

WLOG assume that  $c \geq b \geq a$ . Putting  $a = 0, b = \frac{\sqrt{5} - 1}{2}, c = \frac{\sqrt{5} + 1}{2}$ ; we obtain  $k = 25\sqrt{5}$ . So we have to prove that

$$(a + b + c)^5 \geq 25\sqrt{5}(ab + bc + ca)(b - a)(c - b)(c - a).$$

Note that

$$\begin{aligned} (ab + bc + ca)(bc - ca)(bc - ab) &\leq \frac{1}{27}(ab + bc + ca + bc - ca + bc - ab)^3 \\ &= b^3c^3; \end{aligned}$$

So that we get

$$(ab + bc + ca)(c - a)(b - a) \leq b^2c^2.$$

Hence it is sufficient to check that

$$(a + b + c)^5 \geq 25\sqrt{5}b^2c^2(c - b).$$

Note that, from the AM-GM inequality, we have

$$\begin{aligned} b^2c^2(c - b) &= \left( \left( \frac{\sqrt{5} + 1}{2} \right) b \right)^2 \left( \left( \frac{\sqrt{5} - 1}{2} \right) c \right)^2 (c - b) \\ &\leq \left( \frac{b(\sqrt{5} + 1) + c(\sqrt{5} - 1) + (c - b)}{5} \right)^5 \\ &= \frac{(b + c)^5}{25\sqrt{5}} \leq \frac{(a + b + c)^5}{25\sqrt{5}}. \end{aligned}$$

Equality holds if and only if  $a = 0, b = \frac{\sqrt{5}-1}{2}k, c = \frac{\sqrt{5}+1}{2}k$ , (where  $k$  is a positive constant) and its relevant permutations.  $\square$

### 4.1.3 Remarks

The reason why the calculus approach is not a standalone is that our intuition tells us that the function  $\frac{(a+b+c)^5}{(ab+bc+ca)(c-a)(b-a)(c-b)}$  is decreasing, and we actually need to prove somehow that it's the case. In this problem, we used simple AM-GM, but it is also possible to show with the help of calculus, ie just differentiating wrt  $\min\{a, b, c\}$  and showing the derivative to be negative works. However, in most of the cases, such measures are not necessary. The  $(ab+bc+ca)$  factor created problems due to the presence of  $(ab+ca)$ , which could be removed by the previous method. Now, let's try and tackle  $a^2+b^2+c^2$  in a similar fashion.

## 4.2 Problem 2.

For nonnegative reals  $a, b, c$ , find the best  $k$  satisfying

$$(a+b+c)^5 \geq k(a^2+b^2+c^2)(a-b)(b-c)(c-a)$$

*Tran Quoc Anh, from "Topic of Ngoiavn"*

### 4.2.1 Determination of $k$ and Everything Else

Again, without hesitation, differentiate  $f(t) = \frac{(t+1)^5}{t(t-1)(t^2+1)}$  to get

$$f'(t) = \frac{(t+1)^4(t^4-6t^3+6t^2-6t+1)}{t^2(t-1)^2(t^2+1)^2};$$

The next challenge is to factorise  $t^4-6t^3+6t^2-6t+1$ . Note that we can factor this into  $(t^2+\alpha t+\gamma)\left(t^2+\beta t+\frac{1}{\gamma}\right)$ , After some calculations, we can rightaway get the factorisation as

$$t^4-6t^3+6t^2-6t+1 = \left(t^2-(3+\sqrt{5})t+1\right)\left(t^2-(3-\sqrt{5})t+1\right);$$

Thus the two real roots turn out to be

$$t = \frac{1}{2}\left(3+\sqrt{5} \pm \sqrt{2(5+3\sqrt{5})}\right);$$

By some algebra, we see that  $k = 10\sqrt[4]{5}$ , and the equality case is  $(a, b, c) = (0, \sqrt[4]{5}-1, \sqrt[4]{5}+1)$ . Indeed, for fifth degree, the calculation is devastating.



### 4.2.2 Main Proof

Now we are writing down everything neatly. we need to actually show that

$c \geq b \geq a \geq 0 \implies \frac{(a+b+c)^2}{a^2+b^2+c^2} \geq \frac{(b+c)^2}{b^2+c^2}$ . A somewhat neat way is:

$$\begin{aligned} \frac{(a+b+c)^2}{a^2+b^2+c^2} &= 3 - \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{a^2+b^2+c^2} \geq 3 - \frac{b^2+c^2+(b-c)^2}{b^2+c^2} \\ &= \frac{(b+c)^2}{b^2+c^2}; \end{aligned}$$

So, it is enough to show that

$$(b+c)^5 \geq 10\sqrt[4]{5}(c-b)bc(b^2+c^2).$$

Note that we have,  $bc(b^2+c^2) = \frac{1}{8}((c+b)^4 - (c-b)^4)$ ; so that we may rephrase the last inequality into

$$(b+c)^5 \geq \frac{10\sqrt[4]{5}}{8} \cdot (c-b) \left( (c+b)^4 - (c-b)^4 \right).$$

Due to homogeneity of this inequality, assume  $c+b = 2\sqrt[4]{5}$ , and  $c-b = 2t$ , where  $t \geq 0$ . Then our last inequality simplifies as

$$4 \geq t(5 - t^4).$$

Note that using the AM-GM inequality, we have

$$t(5 - t^4) \leq t(8 - 4t) = 4t(2 - t) \leq (t + 2 - t)^2 = 4.$$

Hence we are done. Equality holds iff  $(a, b, c) \sim \left( (\sqrt[4]{5} + 1), (\sqrt[4]{5} - 1), 0 \right)$  and its relevant permutations.  $\square$

### 4.2.3 Remarks

We adopted the method, and modified it according to our needs of this problem. Indeed, knowing the equality helps, but in cases, the calculations may turn out to be quite long, and awful.

## 5 Other Applications

In this section, we just prove some of the problems that should be done during an exam. This is helpful in demonstrating the power of this method.

### 5.1 Problem 1

For nonnegative  $a, b, c$ , find the best  $k$  such that

$$(a^2 + b^2 + c^2)^3 \geq k(a-b)^2(b-c)^2(c-a)^2$$

*AoPS Forum*

*Solution*

Let us assume without loss of generality that  $c \geq b \geq a$ . Letting  $a = 0, b = \sqrt{5} - 1, c = \sqrt{5} + 1$ ; we get  $k = 27$ . Hence it is sufficient to check the inequality for  $k = 27$ .

However, note that we have  $(a-b)(b-c)(c-a) \leq bc(c-b)$ ; and it is enough to check that

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{(b^2 - 2bc + c^2)(bc)(bc)};$$

Which is obvious from the AM-GM inequality. Equality holds in the original inequality iff  $(a, b, c) \sim (0, \sqrt{5} - 1, \sqrt{5} + 1)$  and all its *symmetric* permutations.

□

### 5.2 Problem 2

For nonnegative  $a, b, c$ , find the best  $k$  such that

$$(a + b + c)^6 \geq k(a^3 + b^3 + c^3)|(a-b)(b-c)(c-a)|$$

*Sayan Mukherjee*

*Solution*

Note that putting  $a = 0, b = -1 + \sqrt{2\sqrt{6} - 3}, c = 1 + \sqrt{2\sqrt{6} - 3}$  leads us to  $k \leq \sqrt{\frac{2}{3}}(2 + \sqrt{6})(2\sqrt{6} - 3)^{\frac{5}{2}}$ . Again, observe that we have

$$\begin{aligned} \frac{(a+b+c)^3}{a^3+b^3+c^3} - \frac{(b+c)^3}{b^3+c^3} &= \frac{3a(-a^2bc + ab^3 + ac^3 + b^4 + b^3c + bc^3 + c^4)}{(b^2 - bc + c^2)(a^3 + b^3 + c^3)} \\ &\geq 0; \end{aligned}$$

Where we assumed  $c \geq b \geq a$ . Thus, it is sufficient to check that

$$(b+c)^6 \geq \sqrt{\frac{2}{3}}(2 + \sqrt{6})(2\sqrt{6} - 3)^{\frac{5}{2}} bc(b^2 - bc + c^2)(c^2 - b^2).$$

Now, let  $c - b = 2t, b + c = 2\sqrt{2\sqrt{6} - 3}$ , so that it is sufficient to check that

$$2^5 (2\sqrt{6} - 3)^{\frac{5}{2}} \geq \sqrt{\frac{2}{3}} (2 + \sqrt{6}) (2\sqrt{6} - 3)^{\frac{5}{2}} \left( \frac{(b+c)^2 - (c-b)^2}{4} \right) \left( \frac{(b+c)^2 + 3(b-c)^2}{4} \right) (c-b);$$

Or,

$$2^5 \geq \sqrt{\frac{2}{3}} (2 + \sqrt{6}) (2\sqrt{6} - 3 - t^2) (2\sqrt{6} - 3 + 3t^2) \cdot 2t;$$

Which can be rewritten into

$$\frac{8\sqrt{6}}{2 + \sqrt{6}} \geq t (2\sqrt{6} - 3 - t^2) (2\sqrt{6} - 3 + 3t^2);$$

Or,

$$t \left( \frac{2\sqrt{6} - 3 - t^2}{2(\sqrt{6} - 2)} \right) \left( \frac{2\sqrt{6} - 3 + 3t^2}{2\sqrt{6}} \right) \leq 1.$$

Observe that, in this form, we may apply the AM-GM inequality to obtain

$$\begin{aligned} \left[ t \left( \frac{2\sqrt{6} - 3 - t^2}{2(\sqrt{6} - 2)} \right) \left( \frac{2\sqrt{6} - 3 + 3t^2}{2\sqrt{6}} \right) \right]^{\frac{2}{5}} &= \sqrt[5]{t^2 \cdot \left( \frac{2\sqrt{6} - 3 - t^2}{2(\sqrt{6} - 2)} \right)^2 \cdot \left( \frac{2\sqrt{6} - 3 + 3t^2}{2\sqrt{6}} \right)^2} \\ &\leq \frac{1}{5} \left[ t^2 + 2 \left( \frac{2\sqrt{6} - 3 - t^2}{2(\sqrt{6} - 2)} \right) + 2 \left( \frac{2\sqrt{6} - 3 + 3t^2}{2\sqrt{6}} \right) \right] \\ &= 1; \end{aligned}$$

And, we are done. Equality holds when  $a = 0, b = -1 + \sqrt{2\sqrt{6} - 3}, c = 1 + \sqrt{2\sqrt{6} - 3}$  and its permutations.  $\square$

## 6 Exercises

### 6.1 Problems

#### Problem 1.

For nonnegative  $a, b, c$ , find the best  $k$  such that

$$(a^2 + b^2 + c^2)^3 \geq k(a + b + c)^3 |(a - b)(b - c)(c - a)|$$

*Sayan Mukherjee*

#### Problem 2.

For three real numbers  $a, b, c \geq 0$ , show that we have the following inequality:

$$(a^3 + b^3 + c^3)^2 \geq k(a + b + c)(ab + bc + ca)|(a - b)(b - c)(c - a)|;$$

Where  $k = \sqrt{\frac{1}{2}(35 + 13\sqrt{13})}$ . *Sayan Mukherjee*

#### Problem 3.

For nonnegative numbers  $a, b, c$ , find the best constant  $k$  such that the following inequality

$$(a^3 + b^3 + c^3)^2 \geq k|(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)|;$$

*Sayan Mukherjee*

#### Problem 4.

For nonnegative numbers  $a, b, c$ , show that we have the following inequality:

$$(a^2 + b^2 + c^2)^3 \geq 6\sqrt{3}(a + b + c)(ab + bc + ca)|(a - b)(b - c)(c - a)|;$$

*Sayan Mukherjee*

#### Problem 5.

For  $a, b, c \geq 0$ , show that we have the following inequality:

$$(a^2 + b^2 + c^2)^2 \geq 4(a + b + c)|(a - b)(b - c)(c - a)|.$$

*Sayan Mukherjee*

## 6.2 Answers

1. Equality at  $\frac{c}{b} = \frac{1}{2} \left( \sqrt{13} - 1 + \sqrt{10 - 2\sqrt{13}} \right)$ ;  $k = \sqrt{\frac{2}{27}} \sqrt{587 - 143\sqrt{13}}$ .  
(Too bad calculations, you need to solve a fourth degree polynomial by the factorisation method mentioned in Section 4.2.1)
2. Equality at  $\frac{c}{b} = \frac{1}{4} \left( \sqrt{13} + 1 + \sqrt{2(\sqrt{13} - 1)} \right)$   
(Somewhat better than the previous problem.)
3. The answer will be the same as Problem 2, just note that  $(a + b + c)(ab + bc + ca) \geq (a + b)(b + c)(c + a)$ ; which means that this one is weaker than Problem 2.
4. Equality occurs in this problem at  $\frac{c}{b} = \frac{\sqrt{3}+1}{\sqrt{2}}$ , and maybe this one looks a bit nicer compared to the rest of the set.
5. An even better one. Equality holds when  $\frac{c}{b} = \sqrt{2} + 1$ .